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Weak convergence to the multiple Stratonovich integral ☆

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Abstract

We have considered the problem of the weak convergence, as ε tends to zero, of the multiple integral processes

$$\left\{\int_0^t \cdots \int_0^t f(t_1,\dots,t_n)\,\mathrm{d}\eta_\varepsilon(t_1)\cdots \mathrm{d}\eta_\varepsilon(t_n), t\in[0,T]\right\}$$

in the space $\mathscr{C}_0([0,T])$, where $f\in L^2([0,T]^n)$ is a given function, and $\{\eta_\varepsilon(t)\}_{\varepsilon>0}$ is a family of stochastic processes with absolutely continuous paths that converges weakly to the Brownian motion. In view of the known results when $n\geqslant 2$ and $f(t_1,\dots,t_n)=1_{\{t_1<t_2<\cdots<t_n\}}$, we cannot expect that these multiple integrals converge to the multiple Itô–Wiener integral of f, because the quadratic variations of the η_ε are null. We have obtained the existence of the limit for any $\{\eta_\varepsilon\}$, when f is given by a multimeasure, and under some conditions on $\{\eta_\varepsilon\}$ when f is a continuous function and when $f(t_1,\dots,t_n)=f_1(t_1)\cdots f_n(t_n)1_{\{t_1<t_2<\cdots<t_n\}}$, with $f_i\in L^2([0,T])$ for any $i=1,\dots,n$. In all these cases the limit process is the multiple Stratonovich integral of the function f.

Keywords: Weak convergence; Multiple Stratonovich integral; Multimeasure; Donsker approximations

1. Introduction

Let Y be a continuous semimartingale and define, for all $t\in[0,T]$, the following iterated Itô integrals:

$$J_k(Y)_t=\begin{cases} Y_t & \text{if } k=1,\\ \displaystyle\int_0^t J_{k-1}(Y)_s\,\mathrm{d}Y_s & \text{for } k\geqslant 2.\end{cases}$$

Suppose that $\{X^\varepsilon\}_{\varepsilon>0}$ is a family of continuous semimartingales that converges weakly to another semimartingale X in the space $\mathscr{C}([0,T])$ of continuous functions on $[0,T]$, as ε tends to zero.

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Then for $m \geqslant 2$, the following statements are equivalent:

$$\mathscr{L}(X^{\varepsilon}, \langle X^{\varepsilon}, X^{\varepsilon}\rangle) \xrightarrow{w} \mathscr{L}(X, \langle X, X\rangle) \quad \text{when } \varepsilon \downarrow 0$$

and

$$\mathscr{L}(J_1(X^{\varepsilon}), \dots, J_m(X^{\varepsilon})) \xrightarrow{w} \mathscr{L}(J_1(X), \dots, J_m(X)) \quad \text{when } \varepsilon \downarrow 0,$$

where the convergence are in the spaces $\mathscr{C}([0,T])^2$ and $\mathscr{C}([0,T])^m$, respectively, and $\langle Y, Y\rangle$ denotes the quadratic variation of the process Y.

The proof of this result is simple: if Y is a continuous semimartingale, we have that

$$J_n(Y)_t = \frac{1}{n!} H_n(Y_t, \langle Y, Y\rangle_t),$$

where $H_n(\cdot, \cdot)$ are the Hermite polynomials in two variables. And thus, $(J_1(Y), \dots, J_m(Y))$ is a continuous functional of $(Y, \langle Y, Y\rangle)$, and conversely, $(Y, \langle Y, Y\rangle)$ is a continuous functional of $(J_1(Y), J_2(Y))$.

Remark 1.1. Avram (1988) has extended this result for semimartingales with trajectories in the space $\mathscr{D}([0,1])$.

This result shows that in order to obtain that $(J_1(X^{\varepsilon}), \dots, J_m(X^{\varepsilon}))$ converges jointly to the multiple Itô integrals with respect to X we need the convergence of X^{ε} to X, but also the convergence of its quadratic variations.

On the other hand, in the literature, there are a lot of important examples in which, the simplest continuous semimartingale, that is, the Brownian motion, can be weakly approximated by processes with absolutely continuous paths. In this case, clearly we do not have the convergence of the quadratic variations to that of the Brownian motion.

It is thus natural to consider the following problem. Let f be a function in the space $L^2([0,T]^n)$, and let $\eta_{\varepsilon} = \{\eta_{\varepsilon}(t);\ t \in [0,T]\}$ be processes with absolutely continuous paths, null at zero and with derivatives in $L^2([0,T])$, that converge weakly to a standard Brownian motion in the space $\mathscr{C}_0([0,T])$ of continuous functions on $[0,T]$ which are null at zero. Consider

$$I_{\eta_{\varepsilon}}(f)_t = \int_0^t \cdots \int_0^t f(t_1, \dots, t_n)\, \mathrm{d}\eta_{\varepsilon}(t_1) \cdots \mathrm{d}\eta_{\varepsilon}(t_n).$$

The aim of this paper is to study the weak convergence of the processes $I_{\eta_{\varepsilon}}(f)$ and, if there is weak convergence, to identify the limit law.

Intuitively, we expect that, when it exists, the limit will be the multiple Stratonovich integral of f, because this integral satisfies the rules of the ordinary differential calculus.

We have first studied when the multiple deterministic integral of a function f with respect to an absolutely continuous function η, as a function of η, admits a continuous extension to the space of all continuous functions $\mathscr{C}_0([0,T])$. In order to have this continuous extension, it is necessary and sufficient that f is given by a multimeasure. In this case, we can prove that $I_{\eta_{\varepsilon}}(f)$ converges weakly to the multiple Stratonovich integral of f.

We have also considered the problem of weak convergence of $I_{\eta_{\varepsilon}}(f)$ for some other classes of functions f that are Stratonovich integrable.

The paper is organized as follows. Section 2 is devoted to some preliminaries on the simple and multiple Stratonovich integrals, and also to give some results on multi-measures. In Section 3 we obtain the characterization of the functions f that define a functional on the Cameron–Martin space possessing a continuous extension to $\mathscr{C}_0([0,T])$, and we check the convergence of $I_{\eta_\varepsilon}(f)$ in this case. In Section 4 we show that, under some conditions on the family $\{\eta_\varepsilon\}_\varepsilon$, the processes $I_{\eta_\varepsilon}(f)$ converge weakly to the multiple Stratonovich integral when the function f is continuous. In the same section we also prove that for the classical Donsker approximations of the Brownian motion process, the last result is also true when f is a factorized Volterra-type function, that is

$$f(t_1,\dots,t_n)=f(t_1)\cdots f(t_n)I_{\{t_1<\cdots<t_n\}}$$

with $f_i\in L^2([0,T])$ for all $i=1,\dots,n$. The last section is an appendix, where we study the integrability in the Stratonovich sense of the classes of functions considered in Section 4, and we prove a technical lemma used in the proof of Theorem 4.3.

Throughout the paper K denotes a positive constant, only depending on the order of the multiple integral and possibly on the function f, whose value may change from one expression to another.

2. Preliminaries

In this work we will consider processes with absolutely continuous paths, defined in a probability space $(\Omega,\mathscr{F},P)$, whose laws are weakly convergent to the Wiener measure. We will also consider a standard Brownian motion, $W=\{W_t,\ t\in[0,T]\}$, defined in a probability space $(\bar{\Omega},\bar{\mathscr{F}},\bar{P})$. The mathematical expectation in these probability spaces will be denoted by E and $\bar{E}$, respectively.

Let π be an arbitrary partition of $[0,T]$, $\pi=\{0=t_0<t_1<\cdots<t_q=T\}$, with the norm $|\pi|=\sup_i(t_{i+1}-t_i)$. We will denote by Δ_i a generic interval determined by π, $\Delta_i=(t_i,t_{i+1}]$, and by $|\Delta_i|$ its length.

Definition. Let $u=\{u_t,\ u\in[0,T]\}$ be a measurable process such that $\int_0^T|u_t|\,\mathrm{d}t<\infty$ a.s. We will say that it is Stratonovich integrable if there exists the limit in $L^2(\Omega)$ of

$$\sum_{i=0}^{q-1}\frac{1}{t_{i+1}-t_i}\left(\int_{t_i}^{t_{i+1}}u_s\,\mathrm{d}s\right)(W(t_{i+1})-W(t_i)),$$

when the norm $|\pi|$ tends to zero. When this limit exists, we will denote it by $\int_0^T u_t\circ\mathrm{d}W_t$. We also denote by $\int_0^t u_s\circ\mathrm{d}W_s$ the Stratonovich integral (if it exists) of $uI_{[0,t]}$, for $t\in[0,T]$.

Definition. Let f be a function in the space $L^2([0,T]^n)$. We will say that it is Stratonovich integrable if there exists the limit in $L^2(\Omega)$ of

$$\sum_{i_1,\dots,i_n}\frac{1}{|\Delta_{i_1}|\cdots|\Delta_{i_n}|}\left(\int_{\Delta_{i_1}\times\cdots\times\Delta_{i_n}}f(t_1,\dots,t_n)\,\mathrm{d}t_1\cdots\mathrm{d}t_n\right)W(\Delta_{i_1})\cdots W(\Delta_{i_n})$$

when the norm $|\pi|$ tends to zero. When this limit exists, we will denote it by $I_n \circ (f)_T$. We will denote by $I_n \circ (f)_t$ the multiple Stratonovich integral (if it exists) of $fI_{[0,t]^n}$, for $t \in [0, T]$.

Definition. Given $f \in L^2([0,T]^m)$ we will say that it possesses trace of order $j \in \{1, \dots, [m/2]\}$ if there exists the limit in $L^2([0,T]^{m-2j})$ of

$$\sum_{i_1,\dots,i_j} \frac{1}{|\Delta_{i_1}| \cdots |\Delta_{i_j}|} \int_{\Delta_{i_1}^2 \times \cdots \times \Delta_{i_j}^2} \tilde{f}(t_1, \dots, t_{2j}, \cdot)\, \mathrm{d}t_1 \cdots \mathrm{d}t_{2j}$$

when the norm $|\pi|$ tends to zero, where $\tilde{f}$ denotes the symmetrization of the function f. When this limit exists we will denote it by $T^j f(\cdot)$.

In this situation we recall the following result proved by Solé and Utzet (1990), known as Hu–Meyer's formula.

Theorem 2.1 (Solé–Utzet). *Let f be a symmetric function in the space $L^2([0,T]^n)$. If there exists its trace of order j for all $j \in \{1, \dots, [n/2]\}$, then f is Stratonovich integrable and*

$$I_n \circ (f) = \sum_{j=0}^{[n/2]} \frac{n!}{(n-2j)!\,j!\,2^j} I^i_{n-2j}(T^j f),$$

where I^i_{n-2j} is the Itô integral of order $n-2j$.

We need also to deal with the notion of multimeasure (see Nualart and Zakai, 1990 for more details).

Definition. Let $(X_1, \mathscr{B}_1), \dots, (X_n, \mathscr{B}_n)$ be measurable spaces. A mapping $\mu : \mathscr{B}_1 \times \cdots \times \mathscr{B}_n \to \mathbb{R}$ is said to be a multimeasure if for every $i \in \{1, \dots, n\}$ and fixed $A_1, \dots, A_{i-1}$, $A_{i+1}, \dots, A_n$ with $A_j \in \mathscr{B}_j$ for all $j \in \{1, \dots, n\} \setminus \{i\}$, $\mu(A_1, \dots, A_{i-1}, F, A_{i+1}, \dots, A_n)$ is a finite signed measure in the variable $F \in \mathscr{B}_i$, namely, μ is the difference of two positive finite measures in F.

Let $\{A_1^k, \dots, A_M^k\}$ denote a measurable partition of X_k.

Definition. Let μ be a multimeasure on $\mathscr{B}_1 \times \cdots \times \mathscr{B}_n$. The Fréchet variation of μ, FV^n, is defined as

$$\|\mu\|_{\mathrm{FV}^n} = \sup \sum_{i_1,\dots,i_n=1}^{M} \varepsilon_{i_1} \cdot \varepsilon_{i_2} \cdots \varepsilon_{i_n} \mu(A^1_{i_1}, \dots, A^n_{i_n}),$$

where ε_i are 1 or -1, for all $i \in \{1, \dots, n\}$, and the supremum is over $\varepsilon \in \{-1, 1\}^n$, and over all finite partitions of the X_k.

Since μ is a multimeasure, it follows that $\|\mu\|_{\mathrm{FV}^n} < \infty$. The class of multimeasures normed by $\|\cdot\|_{\mathrm{FV}^n}$ will be denoted by F^n and becomes a Banach space under this norm.

Let $f_i \in L^\infty(X_i)$ for all $i \in \{1,\dots,k\}$, then

$$\mu_{f_1,\dots,f_k} = \int_{X_1\times\cdots\times X_k} f_1(t_1)\cdot f_2(t_2)\cdot\cdots\cdot f_k(t_k)\mu(\mathrm{d}t_1,\dots,\mathrm{d}t_k,\cdot) \in F^{n-k},$$

and we have that

$$\|\mu_{f_1,\dots,f_k}\|_{\mathrm{FV}^{n-k}} \leqslant \|f_1\|_\infty \cdots \|f_k\|_\infty \cdot \|\mu\|_{\mathrm{FV}^n}. \tag{1}$$

3. Multiple integrals of functions given by multimeasures

Denote by $\mathscr{H}$ the Cameron–Martin space, that is

$$\mathscr{H} = \left\{ \eta \in \mathscr{C}_0([0,T]) \colon \eta_t = \int_0^t \dot\eta_s \,\mathrm{d}s,\ \dot\eta \in L^2([0,T]) \right\}.$$

We can define for a symmetric function $f \in L^2([0,T]^n)$ the following functional:

$$\varphi_f : \mathscr{H} \to \mathscr{C}_0([0,T])$$

$$\eta \to \varphi_f(\eta)_t = \int_0^t \cdots \int_0^t f(x_1,\dots,x_n)\,\mathrm{d}\eta(x_1)\cdots\mathrm{d}\eta(x_n).$$

A first question related with our problem is the following: how must f be in order that the functional φ_f admits a continuous extension to a functional

$$\bar\varphi_f : \mathscr{C}_0([0,T]) \to \mathscr{C}_0([0,T]).$$

When this extension exists, it will be unique, because $\mathscr{H}$ is dense in $\mathscr{C}_0([0,T])$.

This question is obviously related with our problem. Indeed, if such extension exists, then for all $\{\eta_\varepsilon\}_\varepsilon \subset \mathscr{H}$ that converges weakly to the Brownian motion W in $\mathscr{C}_0([0,T])$, we will have that $I_{\eta_\varepsilon}(f) = \bar\varphi_f(\eta_\varepsilon)$ will be weakly convergent to $\bar\varphi_f(W)$, and we will only need to identify this limit.

Let μ be a multimeasure on $[0,T]^n$, and for any $A \in \mathscr{B}([0,T])$, and $t \in [0,T]$ define $A(t)$ as $A \cap [0,t]$ if $t \notin A$ and $A \cup (t,T]$ if $t \in A$.

We can define another multimeasure $\bar\mu_t$ as

$$\bar\mu_t(A_1,\dots,A_n) = \mu(A_1(t),\dots,A_n(t)).$$

It is easy to see that $\bar\mu_t$ is a multimeasure, and that it has the following property: $\bar\mu_t((t_1,t],(t_2,t],\dots,(t_n,t]) = \mu((t_1,T],(t_2,T],\dots,(t_n,T])$ for all $t_i < t$, $i \in \{1,\dots,n\}$.

Our first result is next theorem.

Theorem 3.1. *The following statements are equivalent*:

(a) *φ_f possesses a continuous extension on $\mathscr{C}_0([0,T])$.*

(b) *There exists a symmetric multimeasure μ on $[0,T]^n$ such that $f(x_1,\dots,x_n) = \mu((x_1,T],\dots,(x_n,T])$ a.e.*

Moreover, if $f(x_1,\dots,x_n) = \mu((x_1,T],\dots,(x_n,T])$ then, the extension of φ_f is given by

$$\bar\varphi_f(\eta)_t = \int_{[0,t]^n} \eta(x_1)\cdots\eta(x_n)\bar\mu_t(\mathrm{d}x_1,\dots,\mathrm{d}x_n).$$

Proof. In the proof of this result we will use some ideas of the work of Nualart and Zakai (1990). In this paper the authors study when the multiple Itô–Wiener integral, defined almost surely in the Wiener space, can be extended continuously to $\mathscr{C}_0([0,T])$. The answer is the same for our problem.

To deduce (a) from (b), assume that $f(x_1,\dots,x_n)=\mu((x_1,T],\dots,(x_n,T])$. Define the following functional:

$$\phi_f:\mathscr{H}\times\cdots\times\mathscr{H}\to\mathscr{C}_0([0,T])$$

$$(\eta_1,\dots,\eta_n)\to\phi_f(\eta_1,\dots,\eta_n)_t=\int_{[0,t]^n}f(x_1,\dots,x_n)\,\mathrm{d}\eta_1(x_1)\cdots\mathrm{d}\eta_n(x_n).$$

Observe that

$$\begin{aligned}\phi_f(\eta_1,\dots,\eta_n)_t&=\int_{[0,t]^n}\mu((x_1,T],\dots,(x_n,T])\,\mathrm{d}\eta_1(x_1)\cdots\mathrm{d}\eta_n(x_n)\\&=\int_{[0,t]^n}\bar{\mu}_t((x_1,t],\dots,(x_n,t])\,\mathrm{d}\eta_1(x_1)\cdots\mathrm{d}\eta_n(x_n)\\&=\int_{[0,t]^n}\eta_1(x_1)\cdots\eta_n(x_n)\bar{\mu}_t(\mathrm{d}x_1,\dots,\mathrm{d}x_n),\end{aligned}$$

where we have integrated by parts.

By using (1), we have that

$$\begin{aligned}\left|\int_{[0,t]^n}\eta_1(x_1)\cdots\eta_n(x_n)\bar{\mu}_t(\mathrm{d}x_1,\dots,\mathrm{d}x_n)\right|&\leqslant\|\bar{\mu}_t\|_{\mathrm{FV}^n}\|\eta_1\|_\infty\cdots\|\eta_n\|_\infty\\&\leqslant\|\mu\|_{\mathrm{FV}^n}\|\eta_1\|_\infty\cdots\|\eta_n\|_\infty.\end{aligned}\qquad(2)$$

The last bound has been obtained using that $\sup_{t\in[0,T]}\|\bar{\mu}_t\|_{\mathrm{FV}^n}\leqslant\|\mu\|_{\mathrm{FV}^n}$, by the definition of the Fréchet variation of μ.

Observe that (2) is also satisfied when $\eta_1,\dots,\eta_n\in\mathscr{C}_0([0,T])$.

Since ϕ_f is multilinear, $\mathscr{H}$ is dense in $\mathscr{C}_0([0,T])$, and $\mathscr{C}_0([0,T])$ is a complete metric space, inequality (2) allows to extend ϕ_f to a continuous functional on $\mathscr{C}_0([0,T])\times\cdots\times\mathscr{C}_0([0,T])$, and this extension is obtained by the standard method of approximation of any point $(\eta_1,\dots,\eta_n)$ in $\mathscr{C}_0([0,T])\times\cdots\times\mathscr{C}_0([0,T])$ by a sequence contained in $\mathscr{H}\times\cdots\times\mathscr{H}$. The image of that point will be the limit of the images of the approximating sequence, as a consequence (by using (2) again), it will be equal to

$$\int_{[0,t]^n}\eta_1(x_1)\cdots\eta_n(x_n)\bar{\mu}_t(\mathrm{d}x_1,\dots,\mathrm{d}x_n).$$

In particular, $\varphi_f(\eta)=\phi_f(\eta,\dots,\eta)$ has a continuous extension that is given by

$$\bar{\varphi}_f(\eta)_t=\int_{[0,t]^n}\eta(x_1)\cdots\eta(x_n)\bar{\mu}_t(\mathrm{d}x_1,\dots,\mathrm{d}x_n).$$

To prove now that (a) implies (b), assume that φ_f possesses a continuous extension on $\mathscr{C}_0([0,T])$. We first prove that the functional

$$\Phi_f : \mathscr{H} \times \cdots \times \mathscr{H} \to \mathbb{R},$$

$$(\eta_1,\ldots,\eta_n) \to \phi_f(\eta_1,\ldots,\eta_n)_T = \int_{[0,T]^n} f(x_1,\ldots,x_n)\,\mathrm{d}\eta_1(x_1)\cdots\mathrm{d}\eta_n(x_n),$$

possesses a continuous and multilinear extension on $\mathscr{C}_0([0,T]) \times \cdots \times \mathscr{C}_0([0,T])$.

Denote by $\bar{\varphi}_f$ the continuous extension of φ_f. By Lemma 2.6 of Nualart and Zakai (1990), we can express the product $x_1 \cdot x_2 \cdot \cdots \cdot x_n$, for all $x_1,\ldots,x_n \in \mathbb{R}$, as a linear combination of polynomials of the type $(\alpha_1 x_1 + \cdots + \alpha_n x_n)^n$, where $\alpha = (\alpha_1,\ldots,\alpha_n)$ is a vector of norm one. More precisely, we can write $x_1 \cdot x_2 \cdot \cdots \cdot x_n = \sum_{k=1}^{k_0} \lambda_k (\alpha_1^k x_1 + \cdots + \alpha_n^k x_n)^n$, where $|\alpha^k| = 1$ and $\lambda_k \in \mathbb{R}$ for all $k = 1,\ldots,k_0$.

From this fact and the symmetry of f, we obtain that for any $\eta_1,\ldots,\eta_n \in \mathscr{H}$

$$\begin{aligned}\Phi_f(\eta_1,\ldots,\eta_n) &= \sum_{k=1}^{k_0} \lambda_k \int_{[0,T]^n} f(x_1,\ldots,x_n)\,\mathrm{d}\eta^k(x_1)\cdots\mathrm{d}\eta^k(x_n)\\ &= \sum_{k=1}^{k_0} \lambda_k \varphi_f(\eta^k)_T,\end{aligned}$$

where $\eta^k = \alpha_1^k \eta_1 + \cdots + \alpha_n^k \eta_n$.

And thus, $\bar{\Phi}_f(\eta_1,\ldots,\eta_n) = \sum_{k=1}^{k_0} \lambda_k \bar{\varphi}_f(\eta^k)$ is a continuous extension of Φ_f, that is also multilinear. So, by the generalized Riesz–Fréchet representation theorem (see Nualart and Zakai, 1990, Theorem 2.1), there exists a multimeasure μ on $[0,T]^n$ such that

$$\bar{\Phi}_f(\eta_1,\ldots,\eta_n) = \int_{[0,T]^n} \eta_1(x_1)\cdots\eta_n(x_n)\mu(\mathrm{d}x_1,\ldots,\mathrm{d}x_n)$$

for all $\eta_1,\ldots,\eta_n \in \mathscr{C}_0([0,T])$.

By integration by parts we obtain that for all $\eta_1,\ldots,\eta_n \in \mathscr{H}$ the last expression is equal to

$$\int_{[0,T]^n} \mu((x_1,T],\ldots,(x_n,T])\,\mathrm{d}\eta_1(x_1)\cdots\mathrm{d}\eta_n(x_n).$$

From this, we deduce that $f(x_1,\ldots,x_n) = \mu((x_1,T],\ldots,(x_n,T])$ a.e. □

Corollary 3.2. *Let $\{\eta_\varepsilon\}_{\varepsilon>0}$ be a family of stochastic processes with trajectories in $\mathscr{H}$, that converges weakly to a standard Brownian motion in the space $\mathscr{C}_0([0,T])$. If there exists a multimeasure μ on $[0,T]^n$ such that $f(x_1,\ldots,x_n) = \mu((x_1,T],\ldots,(x_n,T])$ then $I_{\eta_\varepsilon}(f) = \varphi_f(\eta_\varepsilon)$ converges weakly, as ε goes to zero, to the multiple Stratonovich integral of order n of f, $I_n \circ (f)$, in the space $\mathscr{C}_0([0,T])$.*

Proof. In the proof of Theorem 3.1 we have seen that

$$I_{\eta_\varepsilon}(f) = \varphi_f(\eta_\varepsilon) = \int_{[0,t]^n} \eta_\varepsilon(x_1)\cdots\eta_\varepsilon(x_n)\bar{\mu}_t(\mathrm{d}x_1,\ldots,\mathrm{d}x_n).$$

Since η_ε converges weakly to a standard Brownian motion W, in the space $\mathscr{C}_0([0,T])$, the last process will converge in law to the process given by

$$\bar{\varphi}_f(W)_t = \int_{[0,t]^n} W_{x_1} \cdots W_{x_n} \bar{\mu}_t(\mathrm{d}x_1,\dots,\mathrm{d}x_n).$$

On the other hand, by Propositions 2.4 and 4.2 of Nualart and Zakai (1990), the Stratonovich integral $I_n \circ (f)(t)$ exists. Moreover, using Hu–Meyer's formula (see Theorem 2.1) and expression (3.10) of Nualart and Zakai (1990) we have that

$$I_n \circ (f)(t) = \int_{[0,t]^n} W_{x_1} \cdots W_{x_n} \bar{\mu}_t(\mathrm{d}x_1,\dots,\mathrm{d}x_n). \qquad \square$$

4. Weak convergence to the multiple Stratonovich integral of other classes of functions

When the function f is not given by a multimeasure, we cannot expect that for all families $\{\eta_\varepsilon\}_{\varepsilon>0} \subset \mathscr{H}$ that converge weakly to a Brownian motion we will have weak convergence of the processes $I_{\eta_\varepsilon}(f)$.

Nevertheless, if we assume some conditions on the processes $\{\eta_\varepsilon\}$, we can prove some positive results for the cases where f is a continuous function and for

$$f(x_1,\dots,x_n) = f_1(x_1)\cdots f_n(x_n) I_{\{x_1 \leqslant \cdots \leqslant x_n\}}$$

with $f_i \in L^2([0,T])$ for all $i \in \{1,\dots,n\}$.

In both cases, the function f is Stratonovich integrable and $I_n \circ (f)$ has a version with a.s. continuous paths (see Lemmas A.1 and A.2 in the appendix).

From now on we will write our absolutely continuous processes converging in law (in $\mathscr{C}_0([0,T])$) to the Brownian motion as

$$\eta_\varepsilon(t) = \int_0^t \theta_\varepsilon(s)\,\mathrm{d}s,$$

where θ_ε are measurable processes with trajectories a.s. in $L^1([0,T])$.

4.1. The case of continuous functions

4.1.1. Main result

We introduce the following assumption on the family of processes $\{\theta_\varepsilon\}_{\varepsilon>0}$:

(H) There exists an integer $p \geqslant 2$, an increasing continuous function F and a constant $\alpha > 0$ such that for all $0 \leqslant s \leqslant t \leqslant T$,

$$\sup_{\varepsilon>0} \int_{[s,t]^p \times [0,t]^{p(n-1)}} |E(\theta_\varepsilon(x_1)\cdots\theta_\varepsilon(x_{pn}))| I_{\{x_1 \leqslant \cdots \leqslant x_{pn}\}} \mathrm{d}x_1 \cdots \mathrm{d}x_{pn} \leqslant (F(t)-F(s))^{1+\alpha}.$$

Theorem 4.1. *Let f be a symmetric function in the space $\mathscr{C}([0,T]^n)$ and consider $\{\theta_\varepsilon\}_{\varepsilon>0}$ a family of processes that satisfies condition* (H). *Then, the processes $I_{\eta_\varepsilon}(f)$ converge weakly to the multiple Stratonovich integral of f, $I_n \circ (f)$, in the space $\mathscr{C}_0([0,T])$ when ε tends to zero.*

Proof. We start by proving the tightness. Using the Billingsley criterion (see Billingsley, 1968, Theorem 12.3) it suffices to prove that

$$\sup_{\varepsilon} E[|I_{\eta_\varepsilon}(f)_t - I_{\eta_\varepsilon}(f)_s|^{\beta}] \leqslant K(F(t)-F(s))^{1+\alpha}, \tag{3}$$

where $\alpha, \beta > 0$ and F is an increasing continuous function.

For the integer $p \geqslant 2$ of condition (H), we have that

$$\begin{aligned} &E[|I_{\eta_\varepsilon}(f)_t - I_{\eta_\varepsilon}(f)_s|^p] \\ &\quad \leqslant E\left[\left|\int_{[0,t]^n \setminus [0,s]^n} f(x_1,\ldots,x_n)\theta_\varepsilon(x_1)\cdots\theta_\varepsilon(x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n\right|^p\right] \\ &\quad \leqslant \|f\|_\infty^p K \int_{[s,t]^p \times [0,t]^{p(n-1)}} |E(\theta_\varepsilon(x_1)\cdots\theta_\varepsilon(x_{pn}))| I_{\{x_1 \leqslant \cdots \leqslant x_{pn}\}}\,\mathrm{d}x_1\cdots\mathrm{d}x_{pn} \\ &\quad \leqslant \|f\|_\infty^p K(F(t)-F(s))^{1+\alpha}. \end{aligned}$$

We introduce now the following notation:

$$\begin{aligned} X_t &= I_n \circ (f)_t, \\ X_t^\varepsilon &= I_{\eta_\varepsilon}(f)_t = \int_{[0,t]^n} \theta_\varepsilon(x_1)\cdots\theta_\varepsilon(x_n) f(x_1,\ldots,x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n. \end{aligned}$$

We want to see that the finite-dimensional distributions of X^ε converge weakly to those of X. We must check that for all $t_1,\ldots,t_m \in [0,T]$ and any $h \in \mathscr{C}_b^1(\mathbb{R}^m)$, the expression

$$|E[h(X_{t_1}^\varepsilon,\ldots,X_{t_m}^\varepsilon)] - \bar{E}[h(X_{t_1},\ldots,X_{t_m})]|$$

converges to zero when ε tends to zero.

Define

$$\begin{aligned} X_t^{\varepsilon,\pi} &= \sum_{i_1,\ldots,i_n} \left(\frac{1}{|\Delta_{i_1}|\cdots|\Delta_{i_n}|} \int_{\Delta_{i_1}\times\cdots\times\Delta_{i_n}} I_{[0,t]^n} f(y_1,\ldots,y_n)\,\mathrm{d}y_1\cdots\mathrm{d}y_n \right) \\ &\quad \times \eta_\varepsilon(\Delta_{i_1})\cdots\eta_\varepsilon(\Delta_{i_n}), \\ X_t^\pi &= \sum_{i_1,\ldots,i_n} \left(\frac{1}{|\Delta_{i_1}|\cdots|\Delta_{i_n}|} \int_{\Delta_{i_1}\times\cdots\times\Delta_{i_n}} I_{[0,t]^n} f(y_1,\ldots,y_n)\,\mathrm{d}y_1\cdots\mathrm{d}y_n \right) \\ &\quad \times W(\Delta_{i_1})\cdots W(\Delta_{i_n}), \end{aligned}$$

where Δ_i are the intervals of a partition π of $[0,T]$ containing the points $t_1,\ldots,t_m$. We have that

$$|E[h(X_{t_1}^\varepsilon,\ldots,X_{t_m}^\varepsilon)] - \bar{E}[h(X_{t_1},\ldots,X_{t_m})]| \leqslant I_1 + I_2 + I_3$$

with

$$\begin{aligned} I_1 &= |E[h(X_{t_1}^\varepsilon,\ldots,X_{t_m}^\varepsilon) - h(X_{t_1}^{\varepsilon,\pi},\ldots,X_{t_m}^{\varepsilon,\pi})]|, \\ I_2 &= |E[h(X_{t_1}^{\varepsilon,\pi},\ldots,X_{t_m}^{\varepsilon,\pi})] - \bar{E}[h(X_{t_1}^\pi,\ldots,X_{t_m}^\pi)]|, \\ I_3 &= |\bar{E}[h(X_{t_1}^\pi,\ldots,X_{t_m}^\pi)] - \bar{E}[h(X_{t_1},\ldots,X_{t_m})]|. \end{aligned}$$

We will see that I_1 tends to zero, as $|\pi|\to 0$, uniformly in $\varepsilon>0$. Indeed,

$$\begin{aligned}I_1 &= |E[h(X^{\varepsilon}_{t_1},\dots,X^{\varepsilon}_{t_m}) - h(X^{\varepsilon,\pi}_{t_1},\dots,X^{\varepsilon,\pi}_{t_m})]|\\ &\leqslant K\max_j E|X^{\varepsilon}_{t_j} - X^{\varepsilon,\pi}_{t_j}|\\ &\leqslant K\max_j (E(X^{\varepsilon}_{t_j} - X^{\varepsilon,\pi}_{t_j})^2)^{1/2}.\end{aligned}$$

On the other hand,

$$E(X^{\varepsilon}_{t_j} - X^{\varepsilon,\pi}_{t_j})^2 = E\left(\int_{[0,t_j]^n}\theta_\varepsilon(x_1)\cdots\theta_\varepsilon(x_n)g(x_1,\dots,x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n\right)^2,$$

where $g(x_1,\dots,x_n) = f(x_1,\dots,x_n) - f^{\pi}(x_1,\dots,x_n)$ and

$$\begin{aligned}f^{\pi}(x_1,\dots,x_n) = \sum_i \frac{1}{|\Delta_{i_1}|\cdots|\Delta_{i_n}|}&\int_{\Delta_{i_1}\times\cdots\times\Delta_{i_n}} f(y_1,\dots,y_n)\,\mathrm{d}y_1\cdots\mathrm{d}y_n\\ &\times I_{\Delta_{i_1}\times\cdots\times\Delta_{i_n}}(x_1,\dots,x_n).\end{aligned}$$

So, we can bound $E(X^{\varepsilon}_{t_j} - X^{\varepsilon,\pi}_{t_j})^2$ by

$$\left(E\left|\int_{[0,t_j]^n}\theta_\varepsilon(x_1)\cdots\theta_\varepsilon(x_n)g(x_1,\dots,x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n\right|^p\right)^{2/p} \leqslant \|g\|^2_\infty K,$$

by using the same kind of arguments as in the proof of inequality (3). And this last expression goes to zero when $|\pi|$ tends to zero because f^{π} converges in $L^{\infty}([0,T]^n)$ to f, due to the continuity of f.

We have also that, for a fixed partition π,

$$I_2 = |E[h(X^{\varepsilon,\pi}_{t_1},\dots,X^{\varepsilon,\pi}_{t_m})] - \bar{E}[h(X^{\pi}_{t_1},\dots,X^{\pi}_{t_m})]|$$

converges to zero when ε tends to zero because $\mathscr{L}(X^{\varepsilon,\pi}_{t_1},\dots,X^{\varepsilon,\pi}_{t_m}) \xrightarrow{w} \mathscr{L}(X^{\pi}_{t_1},\dots,X^{\pi}_{t_m})$, by the weak convergence of η_ε to the Brownian motion. And finally,

$$\begin{aligned}I_3 &= |\bar{E}[h(X^{\pi}_{t_1},\dots,X^{\pi}_{t_m})] - \bar{E}[h(X_{t_1},\dots,X_{t_m})]|\\ &\leqslant K\max_j \bar{E}|X^{\pi}_{t_j} - X_{t_j}|,\end{aligned}$$

that becomes arbitrarily small by taking $|\pi|$ small enough, because $X^{\pi}_t \xrightarrow{L^2(\bar{\Omega})} X_t$ when $|\pi|$ tends to zero, by the definition of the Stratonovich integral. □

4.1.2. *Examples of processes that satisfy condition* (H)

Donsker and Stroock approximations: Recall that

$$\eta_\varepsilon(t) = \int_0^t \theta_\varepsilon(x)\,\mathrm{d}x,$$

where $t\in[0,T]$.

We consider now the case in which θ_ε are the classical kernels appearing in the known Functional Central Limit Theorem (Donsker kernels),

$$\theta_\varepsilon(x) = \frac{1}{\varepsilon}\sum_{k=1}^{\infty}\zeta_k I_{[k-1,k)}\left(\frac{x}{\varepsilon^2}\right),$$

where ζ_k are independent, centered and identically distributed random variables with $E(\zeta_k^2)=1$.

Or θ_ε are the kernels introduced by Stroock (1982),

$$\theta_\varepsilon(x)=\frac{1}{\varepsilon}(-1)^{N(x/\varepsilon^2)},$$

where $N=\{N(s); s\geqslant 0\}$ is a standard Poisson process.

In order to see that the Donsker kernels satisfy assumption (H), we need to impose some additional requirement on the moments of the random variables ζ_k. More precisely, we assume from now on that $E(\zeta_k^{4n})<\infty$.

To check that both kinds of processes verify condition (H), we will prove a stronger result that will be also useful in Section 4.2 to prove the tightness of another class of processes.

Lemma 4.2. *Let g be a positive function in the space $L^2([0,T])$. There exists a constant K, that only depends on n and on the $L^2([0,T])$-norm of the function g such that, for all $0\leqslant s<t\leqslant T$,*

$$\int_{[s,t]^4\times[0,t]^{4(n-1)}} g(x_1)\cdots g(x_{4n})|E(\theta_\varepsilon(x_1)\cdots\theta_\varepsilon(x_{4n}))|I_{\{x_1\leqslant\cdots\leqslant x_{4n}\}}\,\mathrm{d}x_1\cdots\mathrm{d}x_{4n}$$

$$\leqslant K\left(\int_s^t g^2(x)\,\mathrm{d}x\right)^2,$$

where $\theta_\varepsilon(\cdot)$ are the Donsker or Stroock kernels.

Proof. When θ_ε are the Stroock kernels, we have that

$$\int_{[s,t]^4\times[0,t]^{4(n-1)}} g(x_1)\cdots g(x_{4n})|E(\theta_\varepsilon(x_1)\cdots\theta_\varepsilon(x_{4n}))|I_{\{x_1\leqslant\cdots\leqslant x_{4n}\}}\,\mathrm{d}x_1\cdots\mathrm{d}x_{4n}$$

$$=\int_{[s,t]^4\times[0,t]^{4(n-1)}} g(x_1)\cdots g(x_{4n})\frac{1}{\varepsilon^{4n}}\prod_{j=1}^{2n}\exp\left\{-2\left(\frac{x_{2j}-x_{2j-1}}{\varepsilon^2}\right)\right\}$$

$$\times I_{\{x_1\leqslant\cdots\leqslant x_{4n}\}}\,\mathrm{d}x_1\cdots\mathrm{d}x_{4n}$$

$$\leqslant\left(\int_s^t\int_s^{x_2} g(x_1)g(x_2)\frac{1}{\varepsilon^2}\exp\left\{-2\left(\frac{x_2-x_1}{\varepsilon^2}\right)\right\}\mathrm{d}x_1\,\mathrm{d}x_2\right)^2$$

$$\times\left(\int_0^t\int_0^{x_2} g(x_1)g(x_2)\frac{1}{\varepsilon^2}\exp\left\{-2\left(\frac{x_2-x_1}{\varepsilon^2}\right)\right\}\mathrm{d}x_1\,\mathrm{d}x_2\right)^{2(n-1)}. \tag{4}$$

But

$$\int_s^t\int_s^{x_2} g(x_1)g(x_2)\frac{1}{\varepsilon^2}\exp\left\{-2\left(\frac{x_2-x_1}{\varepsilon^2}\right)\right\}\mathrm{d}x_1\,\mathrm{d}x_2$$

$$\leqslant\frac{1}{2}\int_s^t\int_s^{x_2} g^2(x_1)\frac{1}{\varepsilon^2}\exp\left\{-2\left(\frac{x_2-x_1}{\varepsilon^2}\right)\right\}\mathrm{d}x_1\,\mathrm{d}x_2$$

$$+\frac{1}{2}\int_s^t\int_s^{x_2} g^2(x_2)\frac{1}{\varepsilon^2}\exp\left\{-2\left(\frac{x_2-x_1}{\varepsilon^2}\right)\right\}\,\mathrm{d}x_1\,\mathrm{d}x_2$$

$$\leqslant\frac{1}{2}\int_s^t g^2(x)\,\mathrm{d}x.$$

So, we can bound (4) by

$$K\left(\int_s^t g^2(x)\,\mathrm{d}x\right)^2,$$

where K is a constant only depending on n and on the norm $\|g\|_2$.

When θ_ε are the Donsker kernels, we have that

$$\int_{[s,t]^4\times[0,t]^{4(n-1)}} g(x_1)\cdots g(x_{4n})|E(\theta_\varepsilon(x_1)\cdots\theta_\varepsilon(x_{4n}))|I_{\{x_1\leqslant\cdots\leqslant x_{4n}\}}\,\mathrm{d}x_1\cdots\mathrm{d}x_{4n}$$

$$=\int_{[s,t]^4\times[0,t]^{4(n-1)}} g(x_1)\cdots g(x_{4n})\frac{1}{\varepsilon^{4n}}\left|E\prod_{j=1}^{4n}\left(\sum_{k=1}^{\infty}\zeta_k I_{[k-1,k)}\left(\frac{x_j}{\varepsilon^2}\right)\right)\right|$$

$$\times I_{\{x_1\leqslant\cdots\leqslant x_{4n}\}}\,\mathrm{d}x_1\cdots\mathrm{d}x_{4n}$$

$$\leqslant\sum_{j=1}^{2n}\int_{[s,t]^4\times[0,t]^{4(n-1)}}\prod_{\{l,i_l:i_l\geqslant 2\ \sum_{l=1}^{j} i_l=4n\}}\left(\frac{1}{\varepsilon^{i_l}}\sum_{k=1}^{\infty}|E(\zeta_k^{i_l})|I_{[k-1,k)^{i_l}}\left(\frac{x_1^l}{\varepsilon^2},\ldots,\frac{x_{i_l}^l}{\varepsilon^2}\right)\right.$$

$$\left.\times I_{\{x_1^l\leqslant\cdots\leqslant x_{i_l}^l\}}g(x_1^l)\ldots g(x_{i_l}^l)\right)\,\mathrm{d}x_1^1\ldots\mathrm{d}x_{i_j}^j, \tag{5}$$

where, for all j, $(x_1^1,\ldots,x_{i_1}^1,x_1^2,\ldots,x_{i_j}^j)=(x_1,\ldots,x_{4n})$.

Observe that for all $l\in\{1,\ldots,j\}$, over the integrating set, we have that $x_{i_l}^l-x_1^l<\varepsilon^2$, and so, if $i_l\geqslant 4$

$$\sum_{k=1}^{\infty}I_{[k-1,k)^{i_l}}\left(\frac{x_1^l}{\varepsilon^2},\ldots,\frac{x_{i_l}^l}{\varepsilon^2}\right)I_{\{x_1^l\leqslant\cdots\leqslant x_{i_l}^l\}}$$

$$\leqslant I_{[0,\varepsilon^2)}(x_{i_l}^l-x_1^l)I_{\{x_1^l\leqslant\cdots\leqslant x_{i_l}^l\}}$$

$$\leqslant\prod_{r=1}^{[i_l/2]-1}I_{[0,\varepsilon^2)}(x_{2r}^l-x_{2r-1}^l)I_{\{x_{2r}^l\leqslant x_{2r-1}^l\}}I_{[0,\varepsilon^2)}(x_{i_l}^l-x_{2[i_l/2]-1}^l)I_{\{x_{2[i_l/2]-1}^l\leqslant\cdots\leqslant x_{i_l}^l\}}.$$

From this fact and using that $E(\zeta_k^{4n})<\infty$, expression (5) is less than or equal to

$$K\sum_{\{l,\delta_m:\delta_m\in\{2,3\},\ \sum_{m=1}^{l}\delta_m=4n\}}\int_{[s,t]^4\times[0,t]^{4(n-1)}}\prod_{m=1}^{l}\left(\frac{1}{\varepsilon^{\delta_m}}I_{[0,\varepsilon^2)}(x_{\delta_m}^m-x_1^m)I_{\{x_1^m\leqslant\cdots\leqslant x_{\delta_m}^m\}}\right)$$

$$\times g(x_1^1)\cdots g(x_{\delta_l}^l)\,\mathrm{d}x_1^1\ldots\mathrm{d}x_{\delta_l}^l. \tag{6}$$

But, on the other hand,

$$\int_{[a,b]\times[c,d]} \frac{1}{\varepsilon} g(x_1)\frac{1}{\varepsilon} g(x_2) I_{[0,\varepsilon^2)}(x_2 - x_1) I_{\{x_1 \leqslant x_2\}}\, \mathrm{d}x_1\, \mathrm{d}x_2$$

$$\leqslant \left(\int_{[a,b]\times[c,d]} \frac{1}{\varepsilon^2} g^2(x_1) I_{[0,\varepsilon^2)}(x_2 - x_1) I_{\{x_1 \leqslant x_2\}}\, \mathrm{d}x_1\, \mathrm{d}x_2 \right)^{1/2}$$

$$\times \left(\int_{[a,b]\times[c,d]} \frac{1}{\varepsilon^2} g^2(x_2) I_{[0,\varepsilon^2)}(x_2 - x_1) I_{\{x_1 \leqslant x_2\}}\, \mathrm{d}x_1\, \mathrm{d}x_2 \right)^{1/2}$$

$$= \left(\int_a^b g^2(x)\, \mathrm{d}x \right)^{1/2} \left(\int_c^d g^2(x)\, \mathrm{d}x \right)^{1/2}.$$

And also

$$\int_{[a,b]\times[c,d]\times[e,f]} \frac{1}{\varepsilon^3} g(x_1) g(x_2) g(x_3) I_{[0,\varepsilon^2)}(x_3 - x_1) I_{\{x_1 \leqslant x_2 \leqslant x_3\}}\, \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_3$$

$$\leqslant \int_a^b \frac{1}{\varepsilon} g(x_3) \int_{[\max\{c,x_3-\varepsilon^2\},\min\{x_3,d\}]\times[\max\{e,x_3-\varepsilon^2\},\min\{x_3,f\}]} (1/\varepsilon) g(x_1)(1/\varepsilon) g(x_2)$$

$$\times I_{[0,\varepsilon^2)}(x_2 - x_1) I_{\{x_1 \leqslant x_2\}}\, \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_3.$$

Then, by using the Schwarz inequality and the previous calculation with two variables, this expression is less than or equal to

$$\frac{1}{\varepsilon} \left(\int_a^b g^2(x_3)\, \mathrm{d}x_3 \right)^{1/2} \left(\int_a^b \left(\int_{\max\{c,x_3-\varepsilon^2\}}^{\min\{x_3,d\}} g^2(x_2) \mathrm{d}x_2 \right) \right.$$

$$\left. \times \left(\int_{\max\{e,x_3-\varepsilon^2\}}^{\min\{x_3,f\}} g^2(x_1)\, \mathrm{d}x_1 \right) \mathrm{d}x_3 \right)^{1/2}$$

$$\leqslant \left(\int_a^b g^2(x_3)\, \mathrm{d}x_3 \right)^{1/2} \left(\int_c^d g^2(x_2)\, \mathrm{d}x_2 \int_e^f g^2(x_1)\, \mathrm{d}x_1 \right.$$

$$\left. \times \int_{x_1}^{x_1+\varepsilon^2} \frac{1}{\varepsilon^2}\, \mathrm{d}x_3\, \mathrm{d}x_2\, \mathrm{d}x_1 \right)^{1/2}$$

$$= \left(\int_a^b g^2(x)\, \mathrm{d}x \right)^{1/2} \left(\int_c^d g^2(x)\, \mathrm{d}x \right)^{1/2} \left(\int_e^f g^2(x)\, \mathrm{d}x \right)^{1/2}.$$

And so, expression (6) is bounded by

$$K \left(\int_s^t g^2(x)\, \mathrm{d}x \right)^2.$$

This completes the proof of the lemma. □

Other examples: (1) *Kurtz and Protter kernels.* Another example of processes that satisfy (H) are the following processes, given by Kurtz and Protter (1991), whose integrals converge weakly to the Brownian motion,

$$\theta_\varepsilon(t) = \frac{1}{\varepsilon}\left(W\left(\varepsilon\left(\left[\frac{t}{\varepsilon}\right]+1\right)\right) - W\left(\varepsilon\left[\frac{t}{\varepsilon}\right]\right)\right).$$

Indeed, these processes satisfy the bound given in Lemma 4.2. The proof of this fact can be done following the same arguments as that in the case of Donsker kernels.

(2) *Regularization of the Brownian paths by convolutions.* This kind of approximations is given, for instance, by Ikeda and Watanabe (see Ikeda and Watanabe, 1981, Example VI.7.3).

Let $\phi \in \mathscr{C}^1(\mathbb{R})$ with support on $[0,1]$ and such that $\int_0^1 \phi(x)\,\mathrm{d}x = 1$. Define $\phi_\varepsilon(x) = (1/\varepsilon)\phi(x/\varepsilon)$.

If we consider

$$\eta_\varepsilon(t) = \int_0^t \theta_\varepsilon(r)\,\mathrm{d}r,$$

where

$$\theta_\varepsilon(r) = -\frac{1}{\varepsilon^2}\int_0^\infty W(s)\phi'\left(\frac{s-r}{\varepsilon}\right)\mathrm{d}s,$$

it is easy to see that η_ε converges weakly to W in the space $\mathscr{C}_0([0,T])$. In fact, the convergence is uniform in t a.s. because we have the following alternative expression for η_ε:

$$\eta_\varepsilon(t) = \frac{1}{\varepsilon}\int_0^\infty W(s)\phi\left(\frac{s-t}{\varepsilon}\right)\mathrm{d}s - \frac{1}{\varepsilon}\int_0^\infty W(s)\phi\left(\frac{s}{\varepsilon}\right)\mathrm{d}s.$$

For these processes we can prove that

$$\int_{[s,t]^4\times[0,t]^{4(n-1)}} |E(\theta_\varepsilon(r_1)\cdots\theta_\varepsilon(r_{4n}))| I_{\{r_1\leqslant\cdots\leqslant r_{4n}\}}\,\mathrm{d}r_1\cdots\mathrm{d}r_{4n} \leqslant K(t-s)^2$$

(see Bardina, 1999 for more details).

4.2. The case of factorized Volterra-type functions

Consider

$$f(x_1,\dots,x_l) = f_1(x_1)\cdots f_l(x_l) I_{\{x_1\leqslant\cdots\leqslant x_l\}}, \tag{7}$$

where $f_i \in L^2([0,T])$ for all $i\in\{1,\dots,l\}$, and $\eta_\varepsilon(t) = \int_0^t \theta_\varepsilon(x)\,\mathrm{d}x$ where θ_ε are the Donsker kernels defined in Section 4.1.2.

We will prove in this subsection that

$$I_{\eta_\varepsilon}(f)_t = \int_0^t\cdots\int_0^t f(x_1,\dots,x_l)\,\mathrm{d}\eta_\varepsilon(x_1)\cdots\mathrm{d}\eta_\varepsilon(x_l)$$

converges weakly to the multiple Stratonovich integral of f, $I_l\circ(f)$ in the space $\mathscr{C}_0([0,T])$ and moreover that there is joint convergence of the iterated integrals.

In Lemma A.2, in the appendix, we show that for $n\in\{2,\dots,l\}$ there exist the following iterated Stratonovich simple integrals:

$$Y_n(t)=\int_0^t f_n(u)Y_{n-1}(u)\circ \mathrm{d}W_u,$$

where $Y_1(t)=\int_0^t f_1(u)\,\mathrm{d}W_u$, and that each integral Y_n coincides with the corresponding multiple Stratonovich integral. All of these integrals have a version with continuous paths.

The equality between Y_n and the multiple Stratonovich integral assures the unicity in law of the limit processes.

Theorem 4.3. *Let $f_i\in L^2([0,T])$ for all $i\in\{1,\dots,l\}$, and define*

$$Y_1^\varepsilon(t)=\int_0^t f_1(u)\theta_\varepsilon(u)\,\mathrm{d}u,\quad \textit{and for } n\in\{2,\dots,l\},$$

$$Y_n^\varepsilon(t)=\int_0^t f_n(u)Y_{n-1}^\varepsilon(u)\theta_\varepsilon(u)\,\mathrm{d}u,$$

where θ_ε are the Donsker kernels.

Then,

$$\mathscr{L}(Y_1^\varepsilon,\dots,Y_l^\varepsilon)\xrightarrow{w}\mathscr{L}(Y_1,\dots,Y_l)$$

in the space $(\mathscr{C}_0([0,T]))^l$ when ε tends to zero, where

$$Y_1(t)=\int_0^t f_1(u)\,\mathrm{d}W_u,\quad \textit{and for } n\in\{2,\dots,l\},$$

$$Y_n(t)=\int_0^t f_n(u)Y_{n-1}(u)\circ\mathrm{d}W_u.$$

Remark 4.4. This result is also true when θ_ε are the Stroock kernels defined in Section 4.1.2. We omit the proof to shorten the paper (see Bardina, 1999 for a detailed proof).

In order to prove the previous theorem, the following result will be useful. We denote by $\langle X,Y\rangle$ the quadratic covariation of two continuous martingales X and Y.

Lemma 4.5. *Let $M^{(i)}=\{M_t;\,0\leqslant t\leqslant T\}$, $i=1,\dots,n$ be continuous martingales in the space $(\Omega,\mathscr{F},\{\mathscr{F}_t\},P)$ and suppose that $\langle M^{(i)},M^{(j)}\rangle_t=\int_0^t G_s^{(i)}G_s^{(j)}\,\mathrm{d}s$ for all i and j where $G^{(1)},\dots,G^{(n)}$ are adapted processes of $L^2([0,T]\times\Omega)$. Then, there exists an extension $(\tilde\Omega,\tilde{\mathscr{F}},\tilde P)$ of $(\Omega,\mathscr{F},P)$ where there is defined a standard Brownian motion $W=\{W_t,\tilde{\mathscr{F}}_t;\,0\leqslant t\leqslant T\}$ such that $\bar P$-a.s.*

$$M_t^{(i)}=\int_0^t G_s^{(i)}\,\mathrm{d}W_s,\quad 0\leqslant t\leqslant T,\quad \textit{for all } i=1,\dots,n.$$

Proof. By Theorem 3.4.2 of Karatzas and Shreve (1988), there is an extension $(\tilde\Omega,\tilde{\mathscr{F}},\tilde P)$ of $(\Omega,\mathscr{F},P)$ on which is defined a n-dimensional Brownian motion $B=$

$\{B_t = (B_t^{(1)}, \ldots, B_t^{(n)}), \tilde{\mathscr{F}}_t; 0 \leqslant t \leqslant T\}$ and a matrix $X = \{(X_t^{(i,k)})_{i,k=1}^n, \tilde{\mathscr{F}}_t; 0 \leqslant t \leqslant T\}$ of measurable, adapted processes, such that we have, $\tilde{P}$-a.s., the representations

$$M_t^{(i)} = \sum_{k=1}^n \int_0^t X_s^{(i,k)} \, \mathrm{d}B_s^{(k)}, \quad 1 \leqslant i \leqslant n,$$

$$\langle M^{(i)}, M^{(j)} \rangle_t = \sum_{k=1}^n \int_0^t X_s^{(i,k)} X_s^{(j,k)} \, \mathrm{d}s, \quad 1 \leqslant i, j \leqslant n$$

for all $t \in [0, T]$.

On the other hand, for all $t \in [0, T]$, $\langle M^{(i)}, M^{(j)} \rangle_t = \int_0^t G_s^{(i)} G_s^{(j)} \, \mathrm{d}s$. Then we have that a.e. $XX^{\mathrm{T}} = GG^{\mathrm{T}}$, where $G^{\mathrm{T}} = (G^{(1)} \ldots G^{(n)})$.

From this, we obtain that for any $t \in [0, T]$ there exists a unitary vector U_t such that $X_t = G_t U_t^{\mathrm{T}}$. The proof of this fact is an algebraic exercise, it follows, for instance, using the polar decomposition of a square matrix (see for example Gantmacher, 1998, p. 286). Moreover, we can find a measurable adapted version of U_t. Hence,

$$M_t = (M_t^{(1)} \ldots M_t^{(n)})^{\mathrm{T}} = \int_0^t X_s \, \mathrm{d}B_s$$

$$= \int_0^t G_s U_s^{\mathrm{T}} \, \mathrm{d}B_s = \int_0^t G_s \, \mathrm{d}W_s,$$

where $W_t = \int_0^t U_s^{\mathrm{T}} \, \mathrm{d}B_s$ is a standard one-dimensional Brownian motion. $\square$

Proof of Theorem 4.3. Using the Billingsley criterion (see Billingsley, 1968, Theorem 12.3) in order to prove the tightness it is enough to see that

$$E(Y_n^{\varepsilon}(t) - Y_n^{\varepsilon}(s))^4 \leqslant K \left(\int_s^t g^2(x) \, \mathrm{d}x \right)^2,$$

where $g(x) = \max_{1 \leqslant j \leqslant n} |f_j(x)|$. But using Lemma 4.2,

$$E(Y_n^{\varepsilon}(t) - Y_n^{\varepsilon}(s))^4$$

$$= E \left| \int_{[s,t] \times [0,t]^{n-1}} f_1(x_1) \cdots f_n(x_n) \theta_{\varepsilon}(x_1) \cdots \theta_{\varepsilon}(x_n) I_{\{x_1 \leqslant \cdots \leqslant x_n\}} \, \mathrm{d}x_1 \cdots \mathrm{d}x_n \right|^4$$

$$\leqslant K(n) \int_{[s,t]^4 \times [0,t]^{4(n-1)}} g(x_1) \cdots g(x_{4n}) |E(\theta_{\varepsilon}(x_1) \cdots \theta_{\varepsilon}(x_{4n}))|$$

$$\times I_{\{x_1 \leqslant \cdots \leqslant x_{4n}\}} \, \mathrm{d}x_1 \cdots \mathrm{d}x_{4n}$$

$$\leqslant K \left(\int_s^t g^2(x) \, \mathrm{d}x \right)^2.$$

Denote by P_{ε} the laws of $(Y_1^{\varepsilon}, \ldots, Y_l^{\varepsilon})$ in $\mathscr{C}_0([0, T])^l$. Let $\{P_{\varepsilon_n}\}_n$ be a subsequence of $\{P_{\varepsilon}\}_{\varepsilon}$ (that we will also denote by $\{P_{\varepsilon}\}$) weakly convergent to some probability P. We want to see that the canonical process of $\mathscr{C}_0([0, T])^l$, $(X_1(t), \ldots, X_l(t))$, under the probability P has the same law that $(Y_1(t), \ldots, Y_l(t))$.

Using Lemma 4.5 we will prove that for $n, m \in \{1, \dots, l\}$, under P, the processes

$$X_n(\cdot) - \frac{1}{2}\int_0^{\cdot} f_n(u) f_{n-1}(u) X_{n-2}(u)\,\mathrm{d}u$$

are martingales, with respect to their natural filtration, with quadratic variations and covariations given by

$$\left\langle X_n(\cdot) - \frac{1}{2}\int_0^{\cdot} f_n(u) f_{n-1}(u) X_{n-2}(u)\,\mathrm{d}u, X_m(\cdot) - \frac{1}{2}\int_0^{\cdot} f_m(u) f_{m-1}(u) X_{m-2}(u)\,\mathrm{d}u \right\rangle_t$$

$$= \int_0^t f_n(u) X_{n-1}(u) f_m(u) X_{m-1}(u)\,\mathrm{d}u, \tag{8}$$

where we define $X_{-1} \equiv 0$ and $X_0 \equiv 1$.

To see that under P the processes X_n with their correction terms are martingales with respect to their natural filtration, we will prove that, for any $s_1 \leqslant s_2 \leqslant \cdots \leqslant s_r < s < t$ and for any bounded continuous function $\varphi : \mathbb{R}^{n \times r} \to \mathbb{R}$,

$$E_P\left[\bar{\varphi}\left((X_n(t) - X_n(s)) - \frac{1}{2}\int_s^t f_n(u) f_{n-1}(u) X_{n-2}(u)\,\mathrm{d}u\right)\right] = 0,$$

where $\bar{\varphi} = \varphi((X_1(s_1), \dots, X_n(s_1)), \dots, (X_1(s_r), \dots, X_n(s_r)))$.

But since P_ε converges weakly to P and taking into account the uniform integrability seen in the proof of the tightness, we have that

$$\lim_{\varepsilon \to 0} E_{P_\varepsilon}\left[\bar{\varphi}\left((X_n(t) - X_n(s)) - \frac{1}{2}\int_s^t f_n(u) f_{n-1}(u) X_{n-2}(u)\,\mathrm{d}u\right)\right]$$

$$= E_P\left[\bar{\varphi}\left((X_n(t) - X_n(s)) - \frac{1}{2}\int_s^t f_n(u) f_{n-1}(u) X_{n-2}(u)\,\mathrm{d}u\right)\right].$$

So, it suffices to see that

$$E\left[\varphi^\varepsilon\left((Y_n^\varepsilon(t) - Y_n^\varepsilon(s)) - \frac{1}{2}\int_s^t f_n(u) f_{n-1}(u) Y_{n-2}^\varepsilon(u)\,\mathrm{d}u\right)\right]$$

converges to zero as $\varepsilon \downarrow 0$, where $\varphi^\varepsilon = \varphi((Y_1^\varepsilon(s_1), \dots, Y_n^\varepsilon(s_1)), \dots, (Y_1^\varepsilon(s_r), \dots, Y_n^\varepsilon(s_r)))$. But this expression can be writen as $I_1 - I_2$ where

$$I_1 = \int_s^t \int_0^v f_n(v) f_{n-1}(u) E[\theta_\varepsilon(u)\theta_\varepsilon(v)\varphi^\varepsilon Y_{n-2}^\varepsilon(u)]\,\mathrm{d}u\,\mathrm{d}v,$$

$$I_2 = \frac{1}{2}\int_s^t f_n(u) f_{n-1}(u) E[\varphi^\varepsilon Y_{n-2}^\varepsilon(u)]\,\mathrm{d}u.$$

Moreover,

$$\begin{aligned} I_1 &= \int_s^t \int_0^s f_n(v) f_{n-1}(u) E[\theta_\varepsilon(u)\theta_\varepsilon(v)\varphi^\varepsilon Y_{n-2}^\varepsilon(u)]\,\mathrm{d}u\,\mathrm{d}v \\ &\quad + \int_s^t \int_s^v f_n(v) f_{n-1}(u) E[\theta_\varepsilon(u)\theta_\varepsilon(v)\varphi^\varepsilon Y_{n-2}^\varepsilon(u)]\,\mathrm{d}u\,\mathrm{d}v \\ &= I_{1,1} + I_{1,2}. \end{aligned}$$

Observe that $I_{1,1}$ is null except for $s < v < s+\varepsilon^2$ and $s-\varepsilon^2 < u < s$, since $E(\zeta_k)=0$. But in this case, computing in the same way as in the tightness proof,

$$
\begin{aligned}
I_{1,1} &= E[\varphi^\varepsilon(Y_{n-1}^\varepsilon(s) - Y_{n-1}^\varepsilon(s-\varepsilon^2))(\bar{Y}_1^\varepsilon(s+\varepsilon^2) - \bar{Y}_1^\varepsilon(s))] \\
&\leqslant (E[(Y_{n-1}^\varepsilon(s) - Y_{n-1}^\varepsilon(s-\varepsilon^2))^2])^{1/2}(E[(\varphi^\varepsilon)^2]E[(\bar{Y}_1^\varepsilon(s+\varepsilon^2) - \bar{Y}_1^\varepsilon(s))^2])^{1/2} \\
&\leqslant K\left(\int_{s-\varepsilon^2}^{s} g^2(x)\,\mathrm{d}x\right)^{1/2}(E[(\varphi^\varepsilon)^2])^{1/2}\left(\int_{s}^{s+\varepsilon^2} g^2(x)\,\mathrm{d}x\right)^{1/2},
\end{aligned}
$$

where $\bar{Y}_1^\varepsilon(t) = \int_0^t f_n(x)\theta_\varepsilon(x)\,\mathrm{d}x$. And this last expression goes to zero when ε tends to zero.

We will study now $I_{1,2}$. If ε is small enough, $s - s_r > \varepsilon^2$ since $s_r < s$, and then φ^ε is independent of $\theta_\varepsilon(u)$ and $\theta_\varepsilon(v)$. Using this argument and Lemma 5.3 (given in the appendix) we can see that all the terms in the development of $I_{1,2}$ in which appear $E(\zeta_k^j)$, with $j > 2$, converge to zero. Then, we have that $I_{1,2}$ is equal to

$$
\int_s^t \int_s^v f_n(v)f_{n-1}(u)\frac{1}{\varepsilon^2}\left(\sum_{k=1}^{\infty} I_{[k-1,k)^2}\left(\frac{u}{\varepsilon^2},\frac{v}{\varepsilon^2}\right)\right) E[\varphi^\varepsilon Y_{n-2}^\varepsilon(u)]\,\mathrm{d}u\,\mathrm{d}v
$$

plus some terms that converge to zero.

And therefore, except for some terms tending to zero, $I_{1,2} - I_2$ is equal to

$$
\int_s^t f_{n-1}(u)E[\varphi^\varepsilon Y_{n-2}^\varepsilon(u)] \int_u^t \left(f_n(v)\frac{1}{\varepsilon^2}\left(\sum_{k=1}^{\infty} I_{[k-1,k)^2}\left(\frac{u}{\varepsilon^2},\frac{v}{\varepsilon^2}\right)\right)\mathrm{d}v - \frac{1}{2}f_n(u)\right)\mathrm{d}u,
$$

that converges to zero when ε tends to zero since for all $h_1, h_2 \in L^2([0,T])$,

$$
\begin{aligned}
&\int_s^t h_1(u)\int_u^t h_2(v)\frac{1}{\varepsilon^2}\left(\sum_{k=1}^{\infty} I_{[k-1,k)^2}\left(\frac{u}{\varepsilon^2},\frac{v}{\varepsilon^2}\right)\right)\mathrm{d}v\,\mathrm{d}u \\
&\quad = \sum_{k=1}^{\infty}\frac{1}{\varepsilon^2}\int_{(k-1)\varepsilon^2}^{k\varepsilon^2}\left(\int_{(k-1)\varepsilon^2}^{v} h_1(u)I_{[s,t]}(u)du\right) h_2(v)I_{[s,t]}(v)\,\mathrm{d}v
\end{aligned}
$$

and it is easy to see that this expression tends to

$$
\frac{1}{2}\int_s^t h_1(u)h_2(u)\,\mathrm{d}u.
$$

By the same arguments of the martingale property proof, to check (8) it is enough to show that, for any $n,m \in \{1,\dots,l\}$, $s_1 \leqslant s_2 \leqslant \dots \leqslant s_r < s < t$ and for any bounded continuous function $\varphi: \mathbb{R}^{\max\{n,m\}\times r} \longrightarrow \mathbb{R}$,

$$
\begin{aligned}
E\Bigg[\varphi^\varepsilon\Bigg[&\left((Y_n^\varepsilon(t) - Y_n^\varepsilon(s)) - \frac{1}{2}\int_s^t f_n(u)f_{n-1}(u)Y_{n-2}^\varepsilon(u)\,\mathrm{d}u\right) \\
&\times\left((Y_m^\varepsilon(t) - Y_m^\varepsilon(s)) - \frac{1}{2}\int_s^t f_m(u)f_{m-1}(u)Y_{m-2}^\varepsilon(u)\,\mathrm{d}u\right) \\
&- \int_s^t f_n(u)Y_{n-1}^\varepsilon(u)f_m(u)Y_{m-1}^\varepsilon(u)\,\mathrm{d}u\Bigg]\Bigg]
\end{aligned}
$$

converges to zero, where $\varphi^\varepsilon = \varphi((Y_1^\varepsilon(s_1),\dots,Y_{\max\{n,m\}}^\varepsilon(s_1)),\dots,(Y_1^\varepsilon(s_r),\dots,Y_{\max\{n,m\}}^\varepsilon(s_r)))$.

The last expectation is equal to the sum of the following integrals:

$$I_1 = E[\varphi^\varepsilon(Y_n^\varepsilon(t) - Y_n^\varepsilon(s))(Y_m^\varepsilon(t) - Y_m^\varepsilon(s))],$$

$$I_2 = E\left[\varphi^\varepsilon(Y_n^\varepsilon(t) - Y_n^\varepsilon(s))\left(\frac{1}{2}\int_s^t f_m(u)f_{m-1}(u)Y_{m-2}^\varepsilon(u)\,\mathrm{d}u\right)\right],$$

$$I_{2'} = E\left[\varphi^\varepsilon(Y_m^\varepsilon(t) - Y_m^\varepsilon(s))\left(\frac{1}{2}\int_s^t f_n(u)f_{n-1}(u)Y_{n-2}^\varepsilon(u)\,\mathrm{d}u\right)\right],$$

$$I_3 = E\left[\varphi^\varepsilon\left(\frac{1}{2}\int_s^t f_n(u)f_{n-1}(u)Y_{n-2}^\varepsilon(u)\,\mathrm{d}u\right)\left(\frac{1}{2}\int_s^t f_m(u)f_{m-1}(u)Y_{m-2}^\varepsilon(u)\,\mathrm{d}u\right)\right]$$

and

$$I_4 = E\left[\varphi^\varepsilon\int_s^t f_n(u)Y_{n-1}^\varepsilon(u)f_m(u)Y_{m-1}^\varepsilon(u)\,\mathrm{d}u\right].$$

Observe that

$$\begin{aligned} I_1 &= E\left[\varphi^\varepsilon\left(\int_s^t\int_0^{u_2} f_n(u_2)f_{n-1}(u_1)Y_{n-2}^\varepsilon(u_1)\theta_\varepsilon(u_1)\theta_\varepsilon(u_2)\,\mathrm{d}u_1\,\mathrm{d}u_2\right)\right.\\ &\quad\left.\times\left(\int_s^t\int_0^{v_2} f_m(v_2)f_{m-1}(v_1)Y_{m-2}^\varepsilon(v_1)\theta_\varepsilon(v_1)\theta_\varepsilon(v_2)\,\mathrm{d}v_1\,\mathrm{d}v_2\right)\right]\\ &= I_{1,1} + I_{1,1'} + I_{1,2} + I_{1,2'}, \end{aligned}$$

where $I_{1,1}, I_{1,1'}, I_{1,2}, I_{1,2'}$ are the expectation I_1 over $\{v_1 < u_2 < v_2\}$, $\{u_1 < v_2 < u_2\}$, $\{u_2 < v_1 < v_2\}$ and $\{v_2 < u_1 < u_2\}$, respectively.

Using Lemma 5.3 we have that, except for some terms that converge to zero, $I_{1,1} + I_{1,1'} - I_4$ is equal to

$$\begin{aligned} &\int_s^t E[\varphi^\varepsilon Y_{n-1}^\varepsilon(u)Y_{m-1}^\varepsilon(u)]\left(\int_u^t \frac{1}{\varepsilon^2}\left(\sum_k I_{[k-1,k)^2}\left(\frac{u_2}{\varepsilon^2},\frac{v_2}{\varepsilon^2}\right)\right)\right.\\ &\quad\left.\times\,(f_m(u)f_n(v) + f_m(v)f_n(u))\,\mathrm{d}v - f_n(u)f_m(u)\right)\mathrm{d}u, \end{aligned}$$

that converges to zero by the same argument of the martingale property proof.

We can write $I_3 = I_{3,1} + I_{3,1'}$ where

$$I_{3,1} = \frac{1}{4}\int_s^t\int_s^u f_{n-1}(u)f_n(u)f_{m-1}(v)f_m(v)E[\varphi^\varepsilon Y_{n-2}^\varepsilon(u)Y_{m-2}^\varepsilon(v)]\,\mathrm{d}v\,\mathrm{d}u$$

and $I_{3,1'}$ is the equivalent expression by interchanging the roles of n and m.

Observe that

$$\begin{aligned} I_2 &= E\left[\varphi^\varepsilon\left(\int_s^t\int_0^{u_2} f_n(u_2)f_{n-1}(u_1)Y_{n-2}^\varepsilon(u_1)\theta_\varepsilon(u_1)\theta_\varepsilon(u_2)\,\mathrm{d}u_1\,\mathrm{d}u_2\right)\right.\\ &\quad\left.\times\left(\frac{1}{2}\int_s^t f_{m-1}(v)f_m(v)Y_{m-2}^\varepsilon(v)\,\mathrm{d}v\right)\right]\\ &= I_{2,1} + I_{2,2} + I_{2,3}, \end{aligned}$$

where $I_{2,1}, I_{2,2}, I_{2,3}$ are the expectations over $\{v < u_1 < u_2\}$, $\{u_1 < v < u_2\}$ and $\{u_1 < u_2 < v\}$, respectively. And we can also write $I_{2'}$ as the sum of $I_{2,1'}, I_{2,2'}, I_{2,3'}$ that are equal to $I_{2,1}, I_{2,2}, I_{2,3}$, respectively, interchanging the roles of n and m.

Then, except for some terms that converge to zero, $I_{3,1} - I_{2,1}$ is equal to

$$\frac{1}{2}\int_s^t f_{m-1}(v) f_m(v) \int_v^t f_{n-1}(u_1) E[\varphi^\varepsilon Y_{n-2}^\varepsilon(u_1) Y_{m-2}^\varepsilon(v)]$$

$$\times \left(\frac{1}{2} f_n(u_1) - \int_{u_1}^t \frac{1}{\varepsilon^2}\left(\sum_k I_{[k-1,k)^2}\left(\frac{u_2}{\varepsilon^2}, \frac{u_1}{\varepsilon^2}\right)\right) f_n(u_2)\,\mathrm{d}u_2 \right) \mathrm{d}u_1\,\mathrm{d}v,$$

that converges to zero by the same argument of the martingale property proof. This argument shows also the convergence to zero of $I_{3,1'} - I_{2,1'}$.

On the other hand,

$$I_{2,2} = E\left[\varphi^\varepsilon \int_s^t \int_s^{u_2} f_n(u_2) Y_{n-1}^\varepsilon(v)\theta_\varepsilon(u_2) \frac{1}{2} f_{m-1}(v) f_m(v) Y_{m-2}^\varepsilon(v)\,\mathrm{d}v\,\mathrm{d}u_2\right].$$

But the integrand is null except for $u_2 - v < \varepsilon^2$. Then, it is equal to

$$\frac{1}{2} E\left[\varphi^\varepsilon \int_s^t \int_v^{v+\varepsilon^2} f_n(u_2) f_{m-1}(v) f_m(v)\theta_\varepsilon(u_2) Y_{n-1}^\varepsilon(v) Y_{m-2}^\varepsilon(v)\,\mathrm{d}u_2\,\mathrm{d}v\right]$$

$$= \frac{1}{2}\int_s^t (E[(\varphi^\varepsilon)^2] E[(\bar{Y}_1^\varepsilon(v+\varepsilon^2) - \bar{Y}_1^\varepsilon(v))^2])^{1/2} f_{m-1}(v) f_m(v)$$

$$\times (E[(Y_{n-1}^\varepsilon(v) Y_{m-2}^\varepsilon(v))^2])^{1/2}\,\mathrm{d}v$$

$$\leqslant K \int_s^t f_{m-1}(v) f_m(v) \left(\int_v^{v+\varepsilon^2} g^2(x)\,\mathrm{d}x\right)^{1/2} \mathrm{d}v$$

by using the calculations of the tightness proof, where $g(x) = \max_j |f_j(x)|$ and $\bar{Y}_1^\varepsilon(t) = \int_0^t f_n(x)\theta_\varepsilon(x)\,\mathrm{d}x$.

By dominated convergence, the last integral converges to zero. The same argument shows the convergence to zero of $I_{2,2'}$.

And finally, except for some terms that converge to zero by Lemma 5.3, $I_{1,2} - I_{2,3}$ equals

$$\int_s^t E[\varphi^\varepsilon Y_{m-2}^\varepsilon(v_1)(Y_n^\varepsilon(v_1) - Y_n^\varepsilon(s))] f_{m-1}(v_1)$$

$$\times \left(\frac{1}{\varepsilon^2}\int_{v_1}^t \left(\sum_k I_{[k-1,k)^2}\left(\frac{u_2}{\varepsilon^2}, \frac{v_1}{\varepsilon^2}\right)\right) f_m(v_2)\,\mathrm{d}v_2 - \frac{1}{2} f_m(v_1)\right) \mathrm{d}v_1,$$

that converges to zero. We have an analogous expression for $I_{1,2'} - I_{2,3'}$.

This completes the proof of the theorem. □

Remark 4.6. We point out that if we consider

$$\int_{[0,t]^n} \left(\sum{}' f_{j_1}(t_1)\cdots f_{j_n}(t_n)\right) I_{\{t_1 \leqslant \cdots \leqslant t_n\}} \theta_\varepsilon(t_1)\cdots\theta_\varepsilon(t_n)\,\mathrm{d}t_1\cdots\mathrm{d}t_n,$$

where the symbol $\sum'$ denotes an arbitrary finite sum, $f_j \in L^2([0,T])$ for all j, and θ_ε are the Donsker kernels, the proof of the last theorem shows also the convergence of these processes to

$$\int_{[0,t]^n} \left(\sum{}' f_{j_1}(t_1) \cdots f_{j_n}(t_n) \right) I_{\{t_1 \leqslant \cdots \leqslant t_n\}} \circ \mathrm{d}W_{t_1} \circ \cdots \circ \mathrm{d}W_{t_n}.$$

Appendix

Lemma A.1. *Let $f \in \mathscr{C}([0,T]^n)$. Then, $f \cdot I_{[0,t]^n}$ is Stratonovich integrable and the process $I_n \circ (f) = \{I_n \circ (f)_t = I_n \circ (f \cdot I_{[0,t]^n})\}$ has a version with continuous paths.*

Proof. Denote by π a partition of $[0,T]$. Using Theorem 2.1, it suffices to prove that the trace of order j exists for all $j = 1,\ldots,[n/2]$.

But by the continuity of f in $[0,t]^n$, it is easily seen that for all $j = 1,\ldots,[n/2]$ the trace of order j of $f I_{[0,t]^n}$ is equal to

$$\int_{[0,t]^j} \tilde{f}(x_1,x_1,x_2,x_2,\ldots,x_j,x_j,\cdot) I_{[0,t]^{n-2j}}(\cdot)\,\mathrm{d}x_1 \cdots \mathrm{d}x_j,$$

where $\tilde{f}$ is the symmetrization of the function f.

Moreover, since the Hu–Meyer formula (Theorem 2.1) expresses the Stratonovich integral as a sum of the Itô integrals, the existence of a continuous version of the multiple integral follows. □

Lemma A.2. *Consider*

$$f(x_1,\ldots,x_l) = f_1(x_1) \cdots f_l(x_l) I_{\{x_1 \leqslant \cdots \leqslant x_l\}},$$

where $f_i \in L^2([0,T])$ for all $i \in \{1,\ldots,l\}$. Then, there exist the following iterated simple integrals

$$Y_n(t) = \int_0^t f_n(u) Y_{n-1}(u) \circ \mathrm{d}W_u$$

for $n \in \{2,\ldots,l\}$, where $Y_1(t) = \int_0^t f_1(u)\,\mathrm{d}W_u$. Moreover, all of these integrals have a continuous version and Y_l coincides with $I_l \circ (f)$.

Proof. In this proof we will use techniques of the Malliavin calculus. We present only a sketch of the proof (see Bardina, 1999 for a detailed proof). The (standard) notations given in Nualart (1995) will be used for the objects of Malliavin calculus.

When $n = 2$, we must see that $f_2(t)Y_1(t)$ is Stratonovich integrable. We have to prove that there exists the limit in $L^2(\bar{\Omega})$ when $|\pi|$ tends to zero of

$$\sum_{i=0}^{q-1} \frac{1}{t_{i+1} - t_i} \left(\int_{t_i}^{t_{i+1}} f_2(t) Y_1(t)\,\mathrm{d}t \right) (W(t_{i+1}) - W(t_i)),$$

where $\pi = \{0 = t_0 < \cdots < t_n = T\}$ is a partition of $[0,T]$.

But the last expression equals to

$$\delta\left(\sum_{i=0}^{q-1}\frac{1}{t_{i+1}-t_i}\int_{t_i}^{t_{i+1}} f_2(t)Y_1(t)\,\mathrm{d}t\, I_{[t_i,t_{i+1})}(s)\right)$$

$$+\sum_{i=0}^{q-1}\frac{1}{t_{i+1}-t_i}\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}} f_2(t)f_1(s)I_{[0,t]}(s)\,\mathrm{d}s\,\mathrm{d}t$$

$$=A+B.$$

The first term A, converges in $L^2(\bar{\Omega})$ to $\int_0^T f_2(t)Y_1(t)\,\mathrm{d}W_t$ because we can see that $f_2(t)Y_1(t)\in\mathbb{L}^{1,2}$.

On the other hand, by standard arguments it follows that B converges to $\frac{1}{2}\int_0^T f_1(t)f_2(t)\,\mathrm{d}t$ when $|\pi|$ tends to zero.

In general, we have to prove that $f_n(t)Y_{n-1}(t)$ is Stratonovich integrable. We follow in the same way as for $n=2$, and we use induction on n to prove that $f_n(t)Y_{n-1}(t)\in\mathbb{L}^{1,2}$ and that

$$\sum_{i=0}^{q-1}\frac{1}{t_{i+1}-t_i}\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}} f_n(t)D_sY_{n-1}(t)\,\mathrm{d}s\,\mathrm{d}t$$

converges in $L^2(\bar{\Omega})$ to $\frac{1}{2}\int_0^T f_n(t)f_{n-1}(t)Y_{n-2}(t)\,\mathrm{d}t$. We want to see now that Y_l coincides with $I_l\circ(f)$.

In Solé and Utzet (1990) a Fubini's theorem is proved for $l=2$. On the other hand, in Delgado and Sanz-Solé (1995) it is proved that the iterated integral coincides with the multiple integral of processes that can be anticipative, but a smoothness condition for the Skorohod integrals of the traces is needed. Our processes, in general, do not satisfy this smoothness condition.

In our case, using the Fubini theorem between the stochastic and the Lebesgue integrals, we can write the iterated integrals as

$$Y_l(f)(t)=$$

$$\sum_{j=0}^{[l/2]}\frac{1}{2^j}\sum_{\{l_1<\cdots<l_{2j}:\forall r,l_{2r}=l_{2r-1}+1\}} I^i_{l-2j}\left(\int_{[0,t]^j} f(t_1,\ldots,t_l)\Big|_{\substack{t_{l_1}=t_{l_2}=s_1\\ \cdots\\ t_{l_{2j-1}}=t_{l_{2j}}=s_j}}\mathrm{d}s_1\cdots\mathrm{d}s_j\right), \tag{9}$$

where I^i_{l-2j} is the Itô integral of order $l-2j$.

But in this case, we can compute the trace of order j of f. In this computation we need to prove that if $h_i\in L^2([0,T])$ for $i=1,\ldots,n$, then

$$\sum_{i_1,\ldots,i_n}\frac{1}{|\Delta_{i_1}|\cdots|\Delta_{i_n}|}\int_{\Delta^2_{i_1}\times\cdots\times\Delta^2_{i_n}} h_1(x_1)h_2(x_2)\cdots h_{2n}(x_{2n})I_{\{x_1\leqslant\cdots\leqslant x_{2n}\}}\,\mathrm{d}x_1\cdots\mathrm{d}x_{2n}$$

converges to

$$\frac{1}{2^n}\int_{[0,T]^n} h_1(y_1)h_2(y_1)h_3(y_2)h_4(y_2)\cdots h_{2n-1}(y_n)h_{2n}(y_n)I_{\{y_1\leqslant\cdots\leqslant y_n\}}\,\mathrm{d}y_1\cdots\mathrm{d}y_n.$$

Thus, we obtain that the trace of order j is

$$T^j f(\cdot) = \frac{j!(l-2j)!}{l!} \sum_{\{l_1<\cdots<l_{2j}:\forall r, l_{2r}=l_{2r-1}+1\}} \int_{[0,t]^j} f(t_1,\ldots,t_l)\Bigg|_{\substack{t_{l_1}=t_{l_2}=s_1\\ \cdots \\ t_{l_{2j-1}}=t_{l_{2j}}=s_j}} \mathrm{d}s_1\cdots \mathrm{d}s_j.$$

It follows that the right-hand side of (9) has the same terms that the right-hand side of the expression of $I_l \circ (f)$ given in the Hu–Meyer formula (see Theorem 2.1). Hence, the iterated integral coincides with the multiple integral. Moreover, from this fact the existence of a continuous version of the multiple integral process also follows. □

Lemma 5.3. *Let $\{\theta_\varepsilon\}_{\varepsilon>0}$ the Donsker kernels. Then*

(a) *For all positive function $g\in L^2([0,T])$, we have that*

$$\int_{[s,t]\times[0,t]^{(n-1)}} g(x_1)\cdots g(x_n)|E(\theta_\varepsilon(x_1)\cdots\theta_\varepsilon(x_n))| I_{\{x_1\leqslant\cdots\leqslant x_n\}}\,\mathrm{d}x_1\cdots \mathrm{d}x_n$$

$$\leqslant \sum_{j=1}^{[n/2]} \int_{[s,t]\times[0,t]^{(n-1)}} \prod_{\{l,i_l:i_l\geqslant 2\ \sum_{l=1}^{j} i_l=n\}} \left(\frac{1}{\varepsilon^{i_l}} \sum_{k=1}^{\infty} |E(\zeta_k^{i_l})| I_{[k-1,k)^{i_l}}\left(\frac{x_1^l}{\varepsilon^2},\ldots,\frac{x_{i_l}^l}{\varepsilon^2}\right) \right.$$

$$\left. \times\ I_{\{x_1^l\leqslant\cdots\leqslant x_{i_l}^l\}} g(x_1^l)\ldots g(x_{i_l}^l)\right) \mathrm{d}x_1^1\ldots \mathrm{d}x_{i_j}^j,$$

where for all j, $(x_1^1,\ldots,x_{i_1}^1,x_1^2,\ldots\ldots,x_{i_j}^j)=(x_1,\ldots,x_n)$.

(b) *All the terms in the right-hand side of the last expression with $i_l>2$ converge to zero.*

Proof. The first part of the lemma is obtained by emulating estimate (5). To prove part (b) it suffices to see that

$$\int_{[s,t]\times[0,t]^{i_l-1}} g(x_1^l)\cdots g(x_{i_l}^l)\frac{1}{\varepsilon^{i_l}} I_{[0,\varepsilon^2)}(x_{i_l}^l - x_1^l) I_{\{x_1^l\leqslant\cdots\leqslant x_{i_l}^l\}}\,\mathrm{d}x_1^l\cdots \mathrm{d}x_{i_l}^l,$$

converges to zero, when $i_l\geqslant 3$. In order to simplify notation we denote i_l by k. Then,

$$\int_{[s,t]\times[0,t]^{k-1}} g(x_1)\cdots g(x_k)\frac{1}{\varepsilon^k} I_{[0,\varepsilon^2)}(x_k-x_1) I_{\{x_1\leqslant\cdots\leqslant x_k\}}\,\mathrm{d}x_1\cdots \mathrm{d}x_k$$

$$= \frac{1}{\varepsilon^k}\int_s^t g(x_k)\left(\int_{[x_k-\varepsilon^2,x_k]^{k-1}} g(x_1)\cdots g(x_{k-1}) I_{\{x_1\leqslant\cdots\leqslant x_{k-1}\}}\,\mathrm{d}x_1\cdots \mathrm{d}x_{k-1}\right)\mathrm{d}x_k$$

$$= \frac{1}{\varepsilon^k} K \int_s^t g(x_k)\left(\int_{x_k-\varepsilon^2}^{x_k} g(x)\,\mathrm{d}x\right)^{k-1} \mathrm{d}x_k$$

$$\leqslant \frac{1}{\varepsilon^k} K \int_s^t g(x_k)\left(\varepsilon^2\int_{x_k-\varepsilon^2}^{x_k} g^2(x)\,\mathrm{d}x\right)^{(k-1)/2} \mathrm{d}x_k$$

$$= \frac{1}{\varepsilon} K \int_s^t g(x_k)\left(\int_{x_k-\varepsilon^2}^{x_k} g^2(x)\,\mathrm{d}x\right)^{(k-1)/2} \mathrm{d}x_k. \tag{10}$$

Observe that $(k-1)/2 \geqslant 1$. Consider now

$$G(y) = \int_0^y g^2(x)\,\mathrm{d}x.$$

Since G is a continuous function there exists a constant K, that only depends on k and on the norm $\|g\|_2$, such that

$$(G(x_k) - G(x_k - \varepsilon^2))^{(k-1)/2} \leqslant K(G(x_k) - G(x_k - \varepsilon^2)).$$

So, we can bound (10) by $(1/\varepsilon)K \int_s^t g(x_k)(\int_{x_k-\varepsilon^2}^{x_k} g^2(x)\,\mathrm{d}x)\,\mathrm{d}x_k$. But this expression is less than or equal to

$$\frac{1}{\varepsilon} K \left(\int_s^t g^2(x_k)\,\mathrm{d}x_k\right)^{1/2} \left(2\int_s^t \int_{x_k-\varepsilon^2}^{x_k} \int_{x_k-\varepsilon^2}^{x_2} g^2(x_1)g^2(x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_k\right)^{1/2}$$

$$\leqslant \|g\|_2 K \frac{1}{\varepsilon} \left(2\int_0^t \int_0^{x_2} g^2(x_1)g^2(x_2) I_{[0,\varepsilon^2]}(x_2 - x_1) \int_{x_2}^{x_2+\varepsilon^2} \mathrm{d}x_k\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{1/2}$$

$$\leqslant K \left(\int_0^t g^2(x_2) \int_{x_2-\varepsilon^2}^{x_2} g^2(x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{1/2}$$

and the integrand of the last expression is bounded by $g^2(x_2)\int_0^T g^2(x)\,\mathrm{d}x$ that belongs to $L^1([0,T])$, and so, by dominated convergence, it goes to zero. □

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